Theorem: If $f \in R[a,b]$, then f is bounded on $[a,b]$

Proof: Assume that f is unbounded on $[a,b]$ with integral L. Then there exists $\delta > 0$ such that if P is any tagged partition of $[a,b]$ with $\|P\| < \delta$, then we have $|S(f,P) - L| < 1$, which implies that

$|S(f,P)| < |L| + 1$. ------- $*$.

Let $H = \{[x_{i-1}, x_i]\}_{n=1}^{n}$ be a partition with $\|H\| < \delta$.

Since $|f|$ is not bounded on $[a,b]$, then there exists at least one subinterval in H, say $[x_{k-1}, x_k]$ on which $|f|$ is not bounded, if $|f|$ is bounded on each subinterval $[x_{i-1}, x_i]$ by M_i, then it is bounded on $[a,b]$ by $\max\{M_1, \text{------} M_n\}$.

Let $t_k = x_i$ for $i \neq k$ and let $t_k \in [x_{k-1}, x_k]$ such that

$$|f(t_k)(x_k - x_{k-1})| > |L| + 1 + \left|\sum_{i \neq k} f(t_i)(x_i - x_{i-1})\right|$$

From the Inequality $|A+B| \geq |A| - |B|$ we have

$$|S(f,H)| \geq |f(t_k)(x_k - x_{k-1})| - \left|\sum_{i \neq k} f(t_i)(x_i - x_{i-1})\right|$$

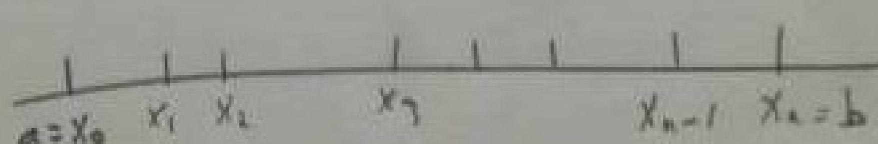
$\geq |L| + 1$ which contradicts with $*$.

<u>Definition:</u> If $I=[a,b]$ is a closed bounded interval in $\mathbb{R}$, then a partition of I is a finite, ordered set $P=(x_0, x_1, \ldots\ldots, x_{n-1}, x_n)$ of points in I such that

$$a=x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

<u>Remark:</u> The points of P are used to divide $I=[a,b]$ into subintervals.

$$I_1=[x_0,x_1],\ I_2=[x_1,x_2], \ldots\ldots, I_n=[x_{n-1},x_n].$$



<u>Remark:</u> We will denote the partition by the notation $P=\{[x_{i-1},x_i]\}_{i=1}^{n}$

Definition: If $I = [a,b]$ is a closed bounded interval in $\mathbb{R}$, then a partition of I is a finite, ordered set $P = (x_0, x_1, \dots, x_{n-1}, x_n)$ of points in I such that

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b.$$

Remark: The points of P are used to divide $I = [a,b]$ into subintervals.

$$I_1 = [x_0, x_1],\ I_2 = [x_1, x_2], \dots, I_n = [x_{n-1}, x_n].$$

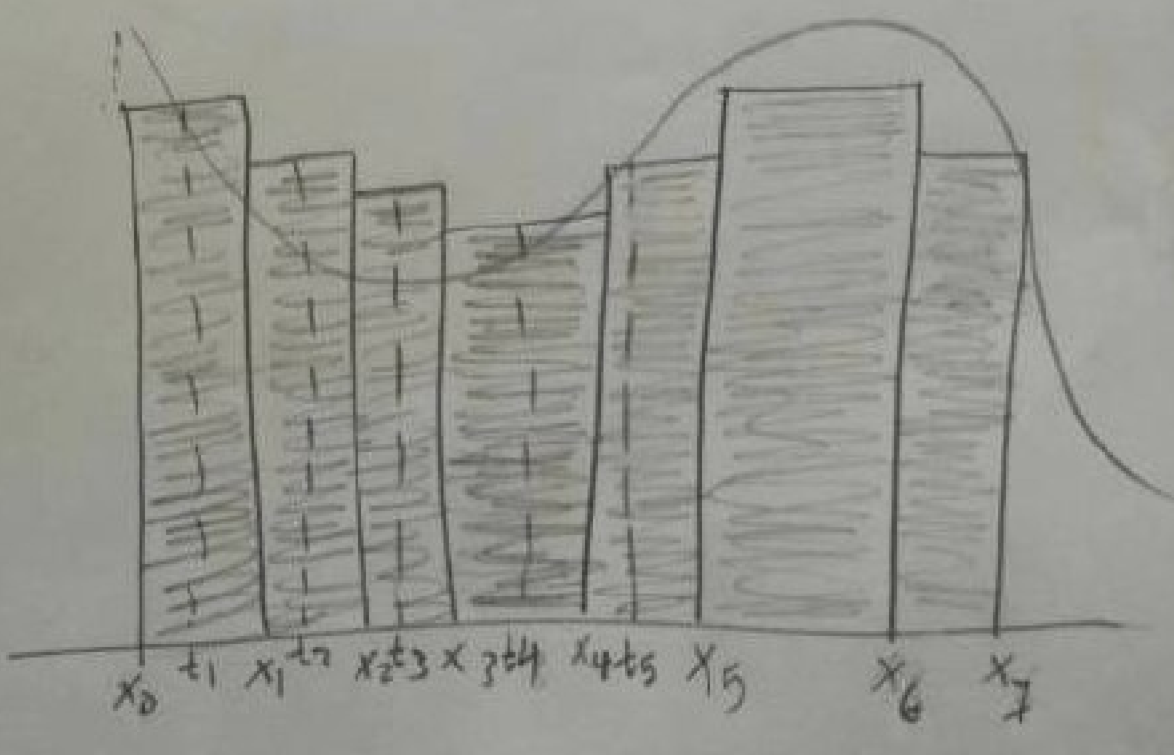
Remark: We will denote the partition by the notation $P = \{[x_{i-1}, x_i]\}_{i=1}^{n}$

<u>Definition</u>: The norm of P to be the number

$$\|P\| = \max\{x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}\}.$$

<u>Remark</u>: The norm of a partition is merely the length of largest subinterval.

<u>Definition</u>: A set of ordered pairs $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ where t_i is a point in the subintervals. is called a tagged partition.



<u>Proof</u>: Assume that L' and L'' both satisfy the definition and let $\epsilon > 0$. Then there exist $\delta'_{\frac{\epsilon}{2}} > 0$ such that if P_1 is any tagged partition with $\|P_1\| < \delta'_{\epsilon/2}$, then $|S(f,P_1) - L'| < \frac{\epsilon}{2}$

Also there exist $\delta''_{\frac{\epsilon}{2}} > 0$ such that if P_2 is tagged partition with $\|P_2\| < \delta''_{\frac{\epsilon}{2}}$. Then $|S(f,P_2) - L''| < \frac{\epsilon}{2}$. Now let

$\delta_\epsilon = \min\{\delta'_{\frac{\epsilon}{2}}, \delta''_{\frac{\epsilon}{2}}\} > 0$ and let P be a tagged partition with $\|P\| < \delta_\epsilon$. Since both $\|P\| < \delta'_{\frac{\epsilon}{2}}$ and $\|P\| < \delta''_{\frac{\epsilon}{2}}$ then

$|S(f,P) - L'| < \frac{\epsilon}{2}$ and $|S(f,P) - L''| < \frac{\epsilon}{2}$

whence it follows from triangle Inequality that

$$\begin{aligned}
|L' - L''| &= |L' - S(f,P) + S(f,P) - L''| \\
&\leq |L' - S(f,P)| + |S(f,P) - L''| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Since ϵ is arbitrary, so $L' = L''$.

Definition: A function $f: [a,b] \longrightarrow \mathbb{R}$ is said to be Riemann integrable on $[a,b]$ if there exist a number $L \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that if P is any tagged partition of $[a,b]$ with $\|P\| < \delta_\epsilon$ then

$$|S(f,P) - L| < \epsilon. \text{ where}$$

$$S(f,P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

The set of all Riemann integrable functions on $[a,b]$ will be denoted by $\mathcal{R}[a,b]$.

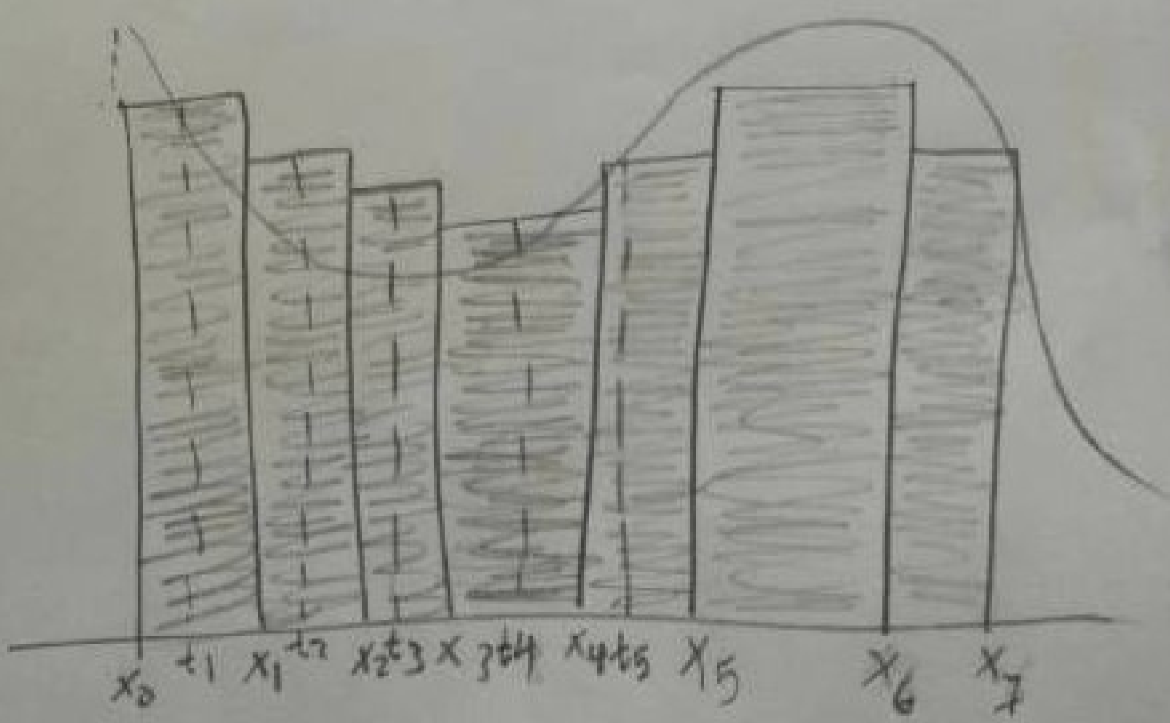
Theorem: If $f \in \mathcal{R}[a,b]$, then the value of the integral uniquely determined.

Definition: The norm of P to be the number

$$\|P\| = \max\{x_1 - x_0, x_2 - x_1, \cdots, x_n - x_{n-1}\}.$$

Remark: The norm of a partition is merely the length of largest subinterval.

Definition: A set of ordered pairs $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ where t_i is a point in the subintervals. is called a tagged partition.



Theorem: If g is Riemann integrable on $[a,b]$ and if $f(x)=g(x)$ except for a finite number of points in $[a,b]$ then f is Riemann integrable and $\int_a^b f = \int_a^b g$.

proof: Let c be a point in the interval and let $L=\int_a^b g$. Assume that $f(x)=g(x)$ for all $x \neq c$. For any tagged partition P, the terms in the two sums $S(f,P)$ and $S(g,P)$ are identical with the exception of at most two terms (in the case that $c=x_i=x_{i-1}$ is an endpoint). Therefore, we have

$$|S(f,P)-S(g,P)| = \left|\sum (f(x_i)-g(x_i))(x_i-x_{i-1})\right| \leq 2(|g(c)|+|f(c)|)\|P\|.$$

Now, given $\epsilon>0$, let $\delta_1>0$ satisfy $\delta_1 < \epsilon/(4(|f(c)|+|g(c)|))$ and let $\delta_2>0$ be such that $\|P\|<\delta_2$ implies $|S(g,P)-L|<\frac{\epsilon}{2}$. Let $\delta=\min\{\delta_1,\delta_2\}$. Then, if $\|P\|<\delta$, then

$$|S(f,P)-L| \leq |S(f,P)-S(g,P)| + |S(g,P)-L| < \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

So the function f is integrable with integral L.

Definition: A function $f: [a,b] \longrightarrow \mathbb{R}$ is said to be Riemann integrable on $[a,b]$ if there exist a number $L \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that if P is any tagged partition of $[a,b]$ with $\|P\| < \delta_\epsilon$ then

$$|S(f,P) - L| < \epsilon. \text{ where}$$

$$S(f,P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

The set of all Riemann integrable functions on $[a,b]$ will be denoted by $\mathcal{R}[a,b]$.

Theorem: If $f \in \mathcal{R}[a,b]$, then the value of the integral uniquely determined.